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# Unitarity, D-brane dynamics and D-brane categories

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Abstract: This is a short nontechnical note summarizing the motivation and results of my recent work on D-brane categories. I also give a brief outline of how this framework can be applied to study the dynamics of topological D-branes and why this has a bearing on the homological mirror symmetry conjecture. This note can be read without any knowledge of category theory.

Keywords: Topological Field Theories, D-branes.

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## 1. Introduction

At the heart of the second superstring revolution one finds a duality in our description of D-brane dynamics. On one hand, D-branes are introduced at the fundamental level as boundary conditions in open string theory, while on the other hand string dualities together with the M-theory interpretation force us to treat them as dynamical objects. There is considerable fuzz surrounding the passage from 'Dirichlet boundary conditions' to 'dynamical objects'. In its most standard incarnation, the argument given takes the following indirect form.

Starting with Dirichlet boundary conditions at the fundamental level, one obtains new open string sectors associated with strings ending on the brane. One next considers the low energy effective action of such *strings*, and identifies it with an effective description of low energy *D-brane* kinematics (the DBI action coupled to background fields). This gives us a low energy description of string fluctuations around the D-brane, and not a description of interactions between D-branes, hence our use of the term *kinematics*.

The DBI action is obviously insufficient for a description of low energy D-brane *dynamics*. Indeed, the effective action of open strings ending on a D-brane describes the low energy dynamics of *strings* with prescribed boundary conditions, but the boundary condition itself is not 'dynamical' in any fundamental way. To describe D-brane interactions, one can resort to studies of string exchange between D-branes, consider the resulting low energy effective action and treat it as an effective description of D-brane interactions (this of course won't give anything interesting for collections of mutually BPS D-branes in type II theories, but there is no reason to restrict to type II or BPS saturated D-branes). Then one can study D-brane interactions through the dynamics of this action. However, effective actions do not give a fundamental (microscopic) description, and the way in which 'boundary conditions' become dynamical is hard to see from such considerations. What, then, is D-brane dynamics ?

A conceptual approach to this issue is afforded by open string field theory. This allows one to answer some dynamical questions at a fundamental level, as demonstrated explicitly by studies of tachyon condensation [2]. In fact, open string tachyon condensation is perhaps the only known example of *true*[1] D-brane dynamics described in a microscopic manner. Through such a process, D-branes are allowed to annihilate, decay, or form bound states. In a certain sense, passage to string field theory performs their 'second quantization'.

There are a few obvious lessons to be learned from studies of tachyon condensation. First, a truly dynamical description of D-branes requires second quantization of strings and off-shell techniques, i.e. string field theory. Second, the notion of D-brane has to be extended.

The second point follows from the observation that the end product of a condensation process is generally not a Dirichlet brane, since it typically cannot be described through boundary conditions on a worldsheet theory. For example, tachyon condensation in superstring compactifications on Calabi-Yau manifolds can produce D-brane composites described by various configurations of bundles and maps [12], for which a direct worldsheet description through a boundary condition is not always available.[2] This implies that, at least in geometrically nontrivial backgrounds, tachyon condensation processes can produce genuinely new objects, distinct from the Dirichlet branes originally considered in the theory.

Moreover, consideration of various condensates in a given background shows that they will generally interact through string exchange. It follows that such condensates behave in many respects as 'abstract D-branes', even though they do not admit a direct description through boundary conditions. This implies that open string theory must be generalized to allow for a description of such objects.

One is thus lead to the task of formulating open string field theory in the presence of 'abstract D-branes'. Since these are not simply boundary conditions, one has to find a structure which allows for their systematic description. The main point of [1] is that the correct structure (at least for the 'associative case' ) is a so-called *dG (or differential graded) category*. This mathematical object arises naturally from constructions based on Dirichlet branes, and — in a slightly less direct manner — also in the case of generalized D-branes (D-brane composites). Moreover, it is showed in [1] that D-brane composite formation can be described as a change of this structure. We reach the following:

**Proposal.** In first nontrivial approximation, D-brane dynamics is described by certain deformations of a dG category.

[1]By true dynamics we mean processes involving interaction and decay of branes, which in particular involve 'second quantization'. In this language, the oscillations of a given D-brane would correspond to its 'first quantization'.

[2]Passage to the derived category as in [20] allows for a representation of *some* such objects as coherent sheaves, some of which can in turn be identified with bundles living on complex submanifolds. However, not every object of the derived category is a coherent sheaf, and not every coherent sheaf can be represented in the second way. Note that we do not claim that such composites cannot be studied with worldheet techniques, only that they cannot be easily described as *boundary conditions*. In a worldsheet approach, such a description will typically require us to deal with the complexities of renormalization group flow.



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By first nontrivial approximation we mean the fact that we only consider tree level dynamics of open strings. Moreover, this approach treats all Dirichlet branes and their condensates on a equal footing ('bootstrap'), though it also opens the way for finding a 'system of generators' which need not be of Dirichlet type. The work of [1], which I shortly review below, is concerned with formulating and exploring some basic consequences of this proposal.

## 2. dG categories on one leg

I now give a short description of the dG category describing usual (i.e. Dirichlet) D-branes. A category [3] is a collection of objects $a$ and sets $\mathrm{Hom}(a,b)$ associated to any ordered pair of objects $a, b$, together with compositions $(u,v) \to uv$ for elements $u$ of $\mathrm{Hom}(b,c)$ and $v$ of $\mathrm{Hom}(a,b)$, where $uv$ belongs to $\mathrm{Hom}(a,c)$. Such compositions are required to be associative, i.e. $(uv)w = u(vw)$, and to admit units $1_a$ (elements of the sets $\mathrm{Hom}(a,a)$ ) such that $u1_a = 1_b v$ for $u, v$ in $\mathrm{Hom}(a,b)$. The objects $a$ and 'morphism sets' $\mathrm{Hom}(a,b)$ can be pretty much anything as long as these requirements are satisfied. Familiar examples are the category **Ens** of sets (with the morphism space between two sets $A, B$ given by all functions from $A$ to $B$ and morphism compositions given by composition of functions), the category of vector spaces **Vct** (with morphisms given by linear maps), and the category $\mathbf{Vect}(X)$ of vector bundles over a manifold $X$ (with morphisms given by bundle morphisms). In all of these cases the elements are some sets (with extra structure) and the morphisms are maps between these sets which preserve the structure (these are so-called 'concrete' categories). A category need not be of this type, however: its objects may not be sets, and its morphisms need not be maps of sets.

As it turns out, Dirichlet branes in an associative oriented open string theory give an example of a (generally non-concrete) category $\mathcal{A}$. This arises by taking Dirichlet branes as objects and the morphism space between two objects to be given by the off-shell state space of open strings stretched between them (figure 1). In general, this space contains the full tower of massive modes, and therefore such morphisms cannot be naturally thought of as maps. The composition of morphisms is given by the string product of [13], which is related to the triple correlator on the disk (figure 1). In an associative string theory, this product is associative off-shell.[3] Moreover, one has units $1_a$, related to the boundary vacua of [14]; these are generalizations of the formal unit of cubic string field theory.

As it happens, the resulting category of Dirichlet branes has some extra structure which reflects the basic data of open string field theory. First, the off-shell state spaces $\mathrm{Hom}(a,b)$ are graded by the ghost degree.[4] If one uses appropriate conventions for the ghost charge, the composition of morphisms preserves this degree, in the sense that the degree of $uv$ is the sum of degrees of $u$ and $v$. In technical language, this means that we have a *graded category*. Another essential ingredient is the worldsheet BRST charge, which

[3]In more general situations, the product need only be associative up to homotopy; this leads to an $A_\infty$ category upon extending the structure discussed in [27] (see also [26]) to the case of backgrounds containing D-branes.

[4]In a topological A/B string theory, this is replaced by the anomalous $U(1)$ charge of the twisted superconformal algebra.



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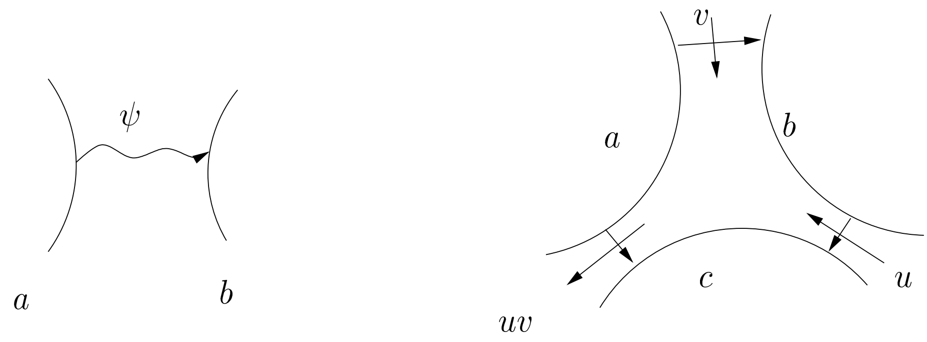


**Figure 1:** Dirichlet branes define a category whose morphisms are off-shell states of oriented open strings stretched between them (left). Compositions of morphisms are given by the string product, which is related to the triple correlator (right).

defines linear operators $Q_{ab}$ on each of the spaces $\mathrm{Hom}(a, b)$; as in [13], these operators act as derivations of the string product. With our conventions, they also have ghost degree $+1$. A graded category endowed with degree one nilpotent[5] operators on its Hom spaces, acting as derivations of morphism compositions, is known as a *differential graded* (dG, for short) category [5, 6]. It follows that the Dirichlet branes of any associative string theory form a dG category. In fact, a complete specification of open string field theory requires some more data, for example a collection of bilinear pairings on morphisms and possibly some complex conjugation operations, which are required to satisfy certain properties. I shall neglect this extra structure in order to simplify the discussion; the bilinear forms are treated in detail in [1].

I should perhaps mention that the dG category considered in this paper will be subject to further identifications. As discussed below, the most important of these is the fact that one must divide by the string field theory gauge group, which is related (in an infinitesimal description) to considering only homotopy equivalence classes of generalized complexes. A slightly larger class of identifications appears when one considers quasiisomorphisms, which can be justified in the BV formalism. The present note ignores these issues, which will treated systematically somewhere else. It should be stressed, however, that these identifications do not generally reduce the complexity of the D-brane category as dramatically as one may hope.[6] Moreover, our main purpose is to describe D-brane *dynamics*, which does not seem to be possible to achieve in a natural manner with on-shell tools (which is what dividing though quasi-isomorphisms would entail). Another issue which is not addressed in this paper is that of the stability of various composites.

## 3. D-brane processes as shifts of the string vacuum

We saw above that Dirichlet branes form a dG category. Does this structure also describe backgrounds containing D-brane condensates ? As we shall see in a moment, the answer is

[5]Remember that an operator is nilpotent if it squares to zero.

[6]The fact that dramatic simplifications seem to occur for open bosonic string theory in $\mathbb{R}^{26}$ is a peculiarity of the extremely simple geometry of that background.

affirmative, though the dG category arising after formation of D-brane composites does not admit a direct construction in terms of string worldsheets (in fact, its description requires consideration of off-shell string dynamics). The nontrivial fact that a dG category can be used to describe both Dirichlet branes and generalized branes arising from condensation of boundary operators is what allows us to view the dG category structure as fundamental.

The basic idea behind this approach is that D-brane composites represent new *boundary sectors*. To understand this, note that a background containing Dirichlet branes can also be described in terms of a 'total boundary state space':

$$\mathcal{H} = \oplus_{a,b}\mathrm{Hom}(a,b)\ , \tag{3.1}$$

endowed with the multiplication induced by morphism compositions. In this approach, one is given a *dG* algebra, i.e. a vector space $\mathcal{H}$ endowed with an associative multiplication and a linear operator $Q = \oplus_{a,b}Q_{ab}$, which squares to zero and acts as a derivation of the product. This is precisely the algebraic framework of [13], expressed with our conventions for the ghost grading. The new input represented by the Dirichlet branes can be described as a decomposition property of the product. Namely, we have a decomposition (3.1) of $\mathcal{H}$ which has the property that the string product vanishes on subspaces of the form $\mathrm{Hom}(b',c) \times \mathrm{Hom}(a,b)$ unless $b' = b$, in which case it maps $\mathrm{Hom}(b,c) \times \mathrm{Hom}(a,b)$ into $\mathrm{Hom}(a,c)$. This decomposition also has the property that it is preserved by the total BRST operator $Q$, i.e. $Q$ preserves each boundary sector $\mathrm{Hom}(a,b)$.

At least formally, a decomposition of $\mathcal{H}$ having these properties is the only piece of data distinguishing the open string field theory of [23] from a theory containing various Dirichlet branes. The component spaces $\mathcal{H}_{ba} = \mathrm{Hom}(a,b)$ of such a decomposition will be called *boundary sectors*. Hence *the underlying D-brane category is determined by the properties of the total string product and total BRST charge.*

This point of view allows us to recover a category structure after formation of D-brane composites takes place. Indeed, such processes are described by condensation of certain boundary/boundary condition changing operators, which correspond to various states $q_{ab}$ in the boundary sectors $\mathrm{Hom}(a,b)$. From the point of view of string field theory, this amounts to giving VEVs $q_{ab}$ to various components $\phi_{ab} \in \mathrm{Hom}(a,b)$ of the total string field $\phi = \oplus_{a,b}\phi_{ab} \in \mathcal{H}$. It follows that the result of a condensation process can be described by the standard device of shifting the string vacuum. Such a shift $\phi \to \phi+q$ preserves the total boundary product, but induces a new BRST operator $Q'$. Moreover, the condition that the new vacuum extremizes the string field action imposes constraints on the allowed shifts $q$. The important observation is that the BRST operator $Q'$ for the shifted vacuum will generally fail to preserve the original boundary sectors $\mathrm{Hom}(a,b)$; this signals the fact that the collection of D-branes in the shifted background has changed, which is exactly what one expects from formation of D-brane composites. One can identify the new boundary sectors (and thus the composite D-branes and the state spaces they determine) by looking for a new decomposition of $\mathcal{H}$ into subspaces, which has the required compatibility properties with respect to the modified BRST operator $Q'$ and the boundary product. This analysis is carried out in [1], with the conclusion that the resulting boundary sectors form a new dG category $\mathcal{A}_q$, the so-called *contraction* of the original category $\mathcal{A}$ along the collection



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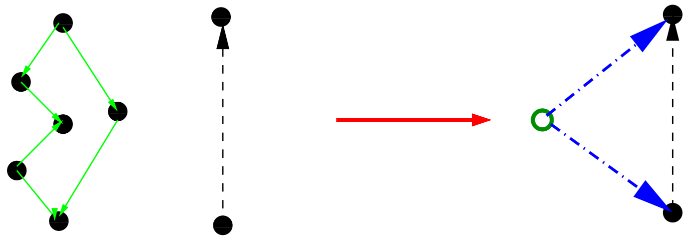

**Figure 2:** Condensation of boundary condition changing operators leads to a new category structure. On the left, the green full lines represent boundary operators which acquire vevs, while the dashed black line is an operator which does not condense. The contracted category on the right is obtained by collapsing the objects connected by the operators which acquire vevs to a single object and building the associated morphism spaces and BRST charges in a systematic manner which is explained in [1]. The new object (hollow circle) on the right represents the D-brane composite formed by condensation of the boundary condition changing operators. Some morphisms between this object and another object of the contracted category are represented by blue dash-and-dot lines. The dashed vertical line represents a morphism which does not change.

of boundary operators $q$ (figure 2). The objects and morphism spaces of this category are given explicitly in [1].

The conclusion is that D-brane composites can once again be described in terms of a dG category, even though they do not generally correspond to Dirichlet branes. Moreover, D-brane composite formation can be described as a change of the dG category structure. This justifies our proposal that the fundamental objects of interest are not Dirichlet branes per se, but rather abstract dG category structures. This amounts to generalizing D-branes to abstract boundary sectors, the latter being specified by decomposition properties of the total boundary product and BRST charge. As discussed above, this generalization is unavoidable if one wishes to allow for D-brane condensation processes, i.e. describe D-branes as truly dynamical objects.

## 4. Unitarity

A basic condition on any description of dynamics is that the underlying space of states be closed under dynamical processes. Since condensation of boundary condition changing operators between Dirichlet branes leads to objects which generally cannot be described through Dirichlet boundary conditions, an open string theory whose boundary sectors are described by Dirichlet branes will typically fail to give a unitary description of D-brane dynamics: its boundary space must be enlarged.

Does a suitable enlargement always exist ? As shown in [1], the answer to this question is affirmative. More precisely, it can be argued that a 'minimal' description which is closed under formation of D-brane composites can be obtained by enlarging the Dirichlet brane category $\mathcal{A}$ to its so-called *quasiunitary cover* $c(\mathcal{A})$. This follows from careful consideration of successive extensions of the original category by adding condensates of Dirichlet branes, condensates of such condensates and so on. The category $c(\mathcal{A})$ can be built explicitly as a category of generalized complexes over $\mathcal{A}$, i.e. sequences of objects $(a_i)$ of $\mathcal{A}$ indexed by

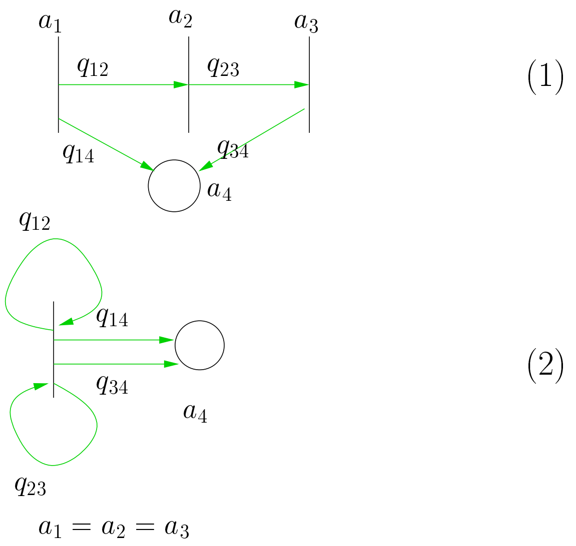


**Figure 3:** Two generalized complexes with four terms. The D-branes $a_1$, $a_2$ and $a_3$ of the first complex are *distinct* objects even though they are translates of each other. That is, D-branes are *not* identified if they differ by a translation (why should they ?); two parallel and non-coincident D-branes are treated as distinct. The second complex contains three identical terms $a_1$, $a_2$ and $a_3$; these correspond to the *same* D-brane $a$. The morphisms $q_{12}$ and $q_{23}$ of the second complex correspond to boundary operators belonging to the sector $\mathrm{Hom}(a, a)$. One can formally think of the second complex as involving repeated condensation of the D-brane $a$ 'with itself'. This is not the same as condensation of a D-brane with some of its translates, which is what is involved in the first complex.

some finite set $I$, together with morphisms $q_{ij} \in Hom^1(a_i, a_j)$ of ghost degree one. These morphisms are subject to the condition:

$$Q_{a_i a_j} q_{ij} + \sum_{k \in I} q_{kj} q_{ik} = 0 \,, \tag{4.1}$$

which is a generalized form of the string equations of motion.

Unitarity of this description follows from the seemingly simple fact that we allow for *sequences* of objects $a_i$, and a sequence may have repetitions. This means that some of the Dirichlet branes $a_i$ may coincide. It is important to realize that we are not talking about deformations of a given D-brane (such as its translations) but literally about identical D-branes in a given sequence. This is illustrated in figure 3.

It is easy to mistake this result for the trivial statement that repetitions in a generalized complex simply amount to condensation of a D-brane with its deformations (such as its translates (if such translates exist in the given background), as shown in part (1) of the figure). This is *not* what we mean, and I hope that the example of figure 3 does something to prevent misunderstanding. 'Condensation of a D-brane with itself' should be viewed as a way of performing deformations of the D-brane. For example, one could have a brane wrapping a special lagrangian cycle in a type II Calabi-Yau compactification and condense a gauge field in order to produce a flat connection on the cycle. Then one could repeat the process by condensing around this new gauge background. This can be viewed as moving



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in the moduli space of flat connections on the cycle, i.e. as performing deformations of the associated D-brane. Finally, the end product of a few such deformations could form a composite with another D-brane (this would correspond to the second complex of figure 3, except that all D-branes should be viewed as wrapped). One could similarly condense the lowest component of a chiral superfield living in the normal bundle, which amounts to deforming the cycle itself. Note, however, that this is a low energy description of such deformations, and in a physically realistic theory one would have to condense higher mass string modes as well, in order to satisfy the string field equations of motion. This gives a notion of 'stringy D-brane deformations' (moduli) which takes into account all $\alpha'$ (and possibly worldsheet instanton) corrections, without making recourse to a description through effective actions. Though figure 3 suggests a flat background and noncompact D-branes, this is for reasons of simplicity only.[7]

Despite such intuitive examples, it should be clear that there is no trivial *universal* justification for introducing generalized complexes, since condensing a D-brane with itself multiple times is harder to understand in, say, a string background based on an abstract conformal field theory.[8] Moreover, our intuitive discussion above is based on low energy considerations, which can be modified rather markedly by stringy effects.

Luckily, a completely general motivation for considering such objects is given by the procedure of recursively including condensates, condensates of condensates and so on in order to obtain a category which is closed under formation of composites. The fact that generalized complexes (together with natural morphisms and morphism compositions) form a dG category which is closed under D-brane composite formation is proved in [1].

Finally, I should probably mention that, though I sometimes freely use the term 'D-brane condensates', I make no claim as to the stability of such composites, which is inaccessible to the simple moduli space analysis employed in [1]. Our composites may in fact be stable, metastable or unstable (='excited states'), but this has no bearing on the unitarity constraint which requires that they all be included in a complete description of the theory. Also note that working with the quasiunitary cover $c(\mathcal{A})$ is similar to a 'bootstrap' approach. What is a 'minimal' set of generators, and what are its properties, is a question for which I do not presently have a good general answer, though I will endeavor to propose some speculations in the conclusions of this note.

## 5. The topological case, twisted complexes and the derived category

I now wish to argue that applying these ideas to the topological B/A models [24, 25, 23] has something to do with derived categories and homological mirror symmetry[17] (this is developed in more detail in [30]). The first step toward making this connec-

[7] As a matter of fact, there are good reasons (coming from the analysis of the A-twisted topological string theory) to believe that A-type D-branes in a type II compactification give a nonassociative category (likely an $A_\infty$ category), due to worldsheet instanton effects. In the simplified discussion above we have neglected this subtlety.

[8] There are other reasons why it is inadvisable to base our arguments on low energy intuition, among them the fact that this would make it difficult to give a conceptual proof of unitarity.



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tion is that Dirichlet branes in a topological string theory are graded objects in a natural manner, a fact which can be related to *extended* deformations of the string vacuum. This means that any Dirichlet brane $a$ has 'formal translates' $a[n]$ for each integer $n$, such that $a[0] = a$. For the topological B model, the objects $a[n]$ can be identified with the graded bundles of [20](whose grading is given by a choice of branch for the argument of the associated central charge [11]). It can be shown that consideration of extended vacuum deformations requires an enlargement of a topological D-brane category $\mathcal{A}$ to its so-called *shift completion* $\tilde{\mathcal{A}}$. This is a dG category with objects $a[n]$ and with a dG structure induced from that of $\mathcal{A}$. Its morphism spaces are given by:

$$\mathrm{Hom}(a[n], b[m]) = \mathrm{Hom}(a, b)[m-n]\,, \tag{5.1}$$

i.e. result by shifts of the grading on the space $\mathrm{Hom}(a, b)$ by $m - n$. It is clear that degree one boundary operators of the shift-completed theory correspond to boundary operators of arbitrary degree between the trivially graded D-branes $a = a[0]$. This captures the intuition that in a topological string theory it should be possible to condense boundary operators of an arbitrary degree, since the grading on the state spaces $\mathrm{Hom}(a, b)$ is somewhat conventional.[9] Condensation of such operators provides the link with extended vacuum deformations.

When combined with our previous results, this observation implies that a topological D-brane category $\mathcal{A}$ should be extended to the quasiunitary cover $\mathcal{B} = c(\tilde{\mathcal{A}})$ of its shift completion $\tilde{\mathcal{A}}$. The objects of $\mathcal{B}$ are so-called *generalized twisted complexes* over $\mathcal{A}$, i.e. sequences of objects of $\mathcal{A}$ together with morphisms of 'arbitrary' degree between them which satisfy a certain version of the string field equations of motion. These objects are closely related to those considered in [6]. Particular examples are the standard complexes:

$$a_1 \stackrel{f_{12}}{\rightarrow} a_2 \stackrel{f_{23}}{\rightarrow} a_3 \ldots \stackrel{f_{n-1n}}{\rightarrow} a_n\,, \tag{5.2}$$

with $f_{i,i+1}$ some degree *zero* morphisms in $\mathrm{Hom}(a_i, a_{i+1})$. In this situation, the underlying constraint on morphisms reduces to the conditions that each $f_{ii+1}$ be BRST closed and that (5.2) is a complex in the category $\mathcal{A}$:

$$\begin{aligned} Q_{a_i a_{i+1}} f_{ii+1} &= 0 \\ f_{i+1i+2} f_{ii+1} &= 0\,. \end{aligned} \tag{5.3}$$

In the case of Calabi-Yau topological B models, whose trivially graded Dirichlet branes $a = a[0]$ are given by holomorphic vector bundles, objects of the type (5.2) reduce to complexes of homolorphic vector bundles and holomorphic maps. Indeed, the morphism space between two bundles $E$ and $F$ (i.e. the off-shell state space of strings stretched from $E$ to $F$) is in this case given by the space $\Omega^{0,*}(E^* \otimes F)$ of forms of types $(0, q)$ valued in

[9]While the degree of the bosonic string field must always be one in our conventions (which follows from the fact [13] that states of non-unitary degree are spurious), such a constraint is not physically fundamental in a topological string theory, whose grading is induced by the anomalous $U(1)$ symmetry of the twisted $N = 2$ superconformal algebra and can therefore be shifted.

the bundle $E^* \otimes F$. The degree of such a form is given by $q$. Hence degree zero states between trivially graded D-branes correspond to (smooth) bundle maps from $E$ to $F$. On the other hand, the BRST charge is given by the Dolbeault operator $\overline{\partial}$ coupled to $E^* \otimes F$ and hence conditions (5.3) reduce to the requirements that the maps $f$ are holomorphic and that they form a complex.

This recovers from a string field theory perspective the complexes considered in [20, 12]. In fact, one can recognize the similarity between some ideas of [20] and our string field arguments, though our approach differs in its insistence on working *off-shell* as much as possible. In our view, characteristics of D-brane dynamics should be *derived* from open string field theory. It is unclear if such a dynamical analysis can be carried out at present for superstring theories, since its formulation seems to be incompletely understood. This is why we retreat to the topological B string, which satisfies the associative framework of [1].

Moreover, we obtain considerably more objects than the standard complexes (5.2). This seems to imply some extension of homological mirror symmetry from the derived category to a certain enlargement. Indeed, it can be shown that, if one starts with the category $\mathcal{A}$ whose elements are holomorphic bundles, and whose morphisms are the spaces $\Omega^{0,*}(E^* \otimes F)$, then the quasiunitary cover of $\tilde{\mathcal{A}}$ is large enough to recover the derived category $D^b(X)$ of the underlying Calabi-Yau manifold $X$, upon performing appropriate localization with respect to quasi-isomorphisms. However, it seems that one can obtain more than $D^b(X)$, which suggests that the original proposal of [17] could be extended to a larger category on the B model side.

A similar discussion can be carried out for the A-model, leading to an enlargement of the category originally considered in [17]. In this case, the situation is complicated by the fact that worldsheet instanton effects violate off-shell associativity of the string product, which implies that the underlying string field theory is described by an $A_\infty$ category [21, 22, 10]. This is further compounded by the fact that the finite radius string vacuum does not satisfy the string equations of motion [22, 15], which implies that the vacuum must be shifted away from its large radius limit. It is possible to generalize our unitarity arguments to the $A_\infty$ case and arrive at a notion of quasiunitary cover of the shift completion of an $A_\infty$ category, which turns out to represent a certain off-shell variant of a proposal made in [17].

Since A-model non-associativity is purely a worldsheet instanton effect, it is not present at large radius and one can describe this limit in more standard mathematical terms. At a large radius point, one in fact has an associative string field theory which fits into the framework of [1]. This theory must be extended to its shift completion as discussed above, which leads to topological D-branes as graded objects. The latter can be identified with the graded lagrangian submanifolds of [9]. The quasiunitary cover of the shift completion can once again be described rather concretely.

## 6. Conclusions and speculations

The essence of our approach is that it allows for a systematic description of D-brane dynamics by exploiting the crucial string field theoretic insight that a correct understanding of string physics requires an off-shell analysis. We believe that a systematic study along these lines can shed light on many crucial issues in D-brane physics such as the relevance of K-theory [7, 8, 18, 19] and the classification of D-brane composites. On the other hand, I would like to propose the *problem of generators*,[10] namely to find a system of generators of the quasiunitary cover $c(\mathcal{A})$ which is minimal in an appropriate sense. More precisely, given a quasiunitary theory $\mathcal{B}$, one would like to find a category $\mathcal{A}_m$ such that $c(\mathcal{A}_m) = \mathcal{B}$ and such that $\mathcal{A}_m$ does not admit a strict subcategory with this property. Intuitively, a minimal system of generators would allow one to think of all objects of $\mathcal{B}$ as composites of the objects of $\mathcal{A}_m$, much in the same way that baryons and mesons are composites of quarks and gluons in QCD. In other words, we are asking if there is some good notion of 'elementary branes'. It is not clear whether a minimal system of generators would be unique in an appropriate sense, and to what extent it would be background dependent. Note also that such generators need not be Dirichlet branes (i.e. need not admit a description through Dirichlet boundary conditions). Finding such minimal generating sets may be the best hope to understand the (typically very large) object $c(\mathcal{A})$ without performing the set of daunting computations that seem to be implied in carrying out our approach for a nontopological string theory. For the topological case of Calabi-Yau B-models, a promising suggestion is offered by the theory of so-called helices (see for example [4]). In this regard, I should perhaps also mention a tantalizing similarity with Matrix theory. Providing a good system of generators essentially amounts to describing all Dirichlet branes and their condensates as composites of some particular objects. From this perspective, Matrix theory amounts to the proposal that D-paticles may suffice, but this now seems to be insufficient in view of difficulties to reconstruct the full spectrum on geometrically nontrivial backgrounds. In our approach, this may be surmounted by the fact that a minimal system of generators may have to include a larger/different class of objects in order to achieve the generation property and some measure of background independence.

The approach outlined above may shed some light on the problem of relating closed and open string moduli. This is intimately connected with the suggestion made in [16, 14] that a *unitary* formulation of open-closed string theory may lead to a certain equivalence between the open and closed sectors, as suggested by the physical interpretation [16] of the formality result proved in [28] for the case of the so-called C model of Cattaneo and Felder [29]. Finally, I believe that a similar analysis could be carried out (at the 2-category level !) for the topological membrane theory formulated in [31]. The relation between the topological membrane of [31] and topological strings seems to be that the first quantization of the membrane gives the *second* quantization of strings. In particular, it is natural to expect that, at least for the topological case, an extension of the work of [31] may provide a description of 'second quantized' D-brane dynamics in terms of the first quantized dynamics of topological membranes. In this sense, M-theory may amount to a description of second quantized open-closed string dynamics through the first quantization of a membrane.[11]

[10] I thank Radu Roiban for a useful conversation on this subject.

[11] And this may suffice, if the poorly understood statement that 'M-theory has no coupling constant' is taken to mean that first quantization of membranes somehow suffices to give a complete description of M-theory.



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